

# A note on energy currents and decay for the wave equation on a Schwarzschild background

Mihalis Dafermos\*      Igor Rodnianski†

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## Abstract

In recent work, we have proven uniform decay bounds for solutions of the wave equation  $\square_g \phi = 0$  on a Schwarzschild exterior, in particular, the uniform pointwise estimate  $|\phi| \leq C v_+^{-1}$  which holds throughout the domain of outer communications, where  $v$  is an advanced Eddington-Finkelstein coordinate,  $v_+ \doteq \max\{v, 1\}$ , and  $C$  is a constant depending on a Sobolev norm of initial data. A crucial estimate in the proof required a decomposition into spherical harmonics. We here give an alternative proof of this estimate not requiring such a decomposition.

In [3], we studied the problem of decay for general solutions  $\phi$  of the equation

$$\square_g \phi = 0 \tag{1}$$

on a Schwarzschild background. The estimates of [3] were obtained by exploiting compatible currents associated with vector field multipliers applied to the energy momentum tensor. (See [2] for a general discussion of such currents.) Understanding the decay properties of solutions of (1) in terms of such energy estimates appears to be a fundamental first step, if one is ever to address the problem of non-linear stability of black hole solutions of the Einstein equations of general relativity.

A crucial role in the results of [3] is played by an energy current  $J_\mu$  related to vector fields of the form  $f(r^*)\partial_{r^*}$ , where  $r^*$  is a Regge-Wheeler coordinate. For the current  $J_\mu$  constructed in [3], the divergence  $K = \nabla^\mu J_\mu$  was shown to be nonnegative upon integration over spheres of symmetry. (The integral of this current over an arbitrary spacetime region  $\mathcal{R}$  was denoted in [3] by  $I^X(\mathcal{R})$ .) The construction of  $J_\mu$  was quite elaborate. In particular, a decomposition into spherical harmonics was required, and a separate definition of  $f_\ell$  was made for each spherical harmonic, characterized by a non-negative integer  $\ell$ . These currents were then summed to obtain a total current. As  $\ell \rightarrow \infty$ , the unique

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\*University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB United Kingdom

†Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544 United States

vanishing point of  $f_\ell$  approached the *photon sphere*  $r = 3M$ . In this sense, the degeneration of the current was seen to be connected to the presence of trapped null geodesics. This degeneration is known to be an essential feature in view of well-known arguments from geometric optics.

With an eye towards possible future applications to problems involving perturbed spacetimes, it is desirable for methods which avoid altogether the use of spherical harmonics. In this short paper, we indeed construct a current  $J_\mu$  with the required non-negativity properties for its divergence  $K = \nabla^\mu J_\mu$ , without recourse to spherical harmonics. See Propositions 3.1, 4.1 and 4.2. The current and its divergence both depend on the 2-jet of the solution  $\phi$ . This current can be substituted in the arguments of [3], completely removing references to spherical harmonics from the proof. See Theorem 6.1 of Section 6 for the basic estimate.

Numerical evidence for the existence of currents with the nonnegativity property achieved here has been presented recently in [1].

One should note finally that an easy perturbation argument shows that the current  $J_\mu$  defined above can also be applied to the study of (1) on Schwarzschild-de Sitter, at least for small  $M\sqrt{\Lambda}$ , thus removing references to spherical harmonics from the proof of Theorem 1.1 of [4] for this case. See Section 7.

## 1 Energy current templates from vector fields

Our notation follows closely that of our [4]. Recall that in the domain of outer communications  $\mathcal{D}$  the Schwarzschild metric can be written explicitly in a coordinate system  $(t, r)$ :

$$g = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2},$$

with coordinate range  $(\infty, \infty) \times (2M, \infty)$ . In what follows we set

$$\mu = \frac{2M}{r}, \quad r^* = r + 2M \ln(r - 2M) - 3M - 2M \ln M,$$

and we let  $'$  denote the derivative with respect to  $r^*$ . Recall that (in our conventions) Regge-Wheeler coordinates  $(t, r^*)$  are related to Eddington-Finkelstein coordinates  $(u, v)$  by the formulas

$$t = v + u, \quad r^* = v - u.$$

In this paper,  $\phi$  will always denote a solution to the wave equation (1) on maximally extended Schwarzschild  $(\mathcal{M}, g)$  which is  $H^2$  on spheres of symmetry and such that  $\nabla\phi$  is in  $L^2$  on spheres of symmetry.<sup>1</sup> Given  $\phi$ , let  $T_{\mu\nu}(\phi)$  denote

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<sup>1</sup>We require this regularity for we shall give integral bounds on spheres of symmetry. By integrating these bounds in spacetime regions, one can apply these results to solutions of (1) with locally  $H^2$  initial data.

the energy momentum tensor

$$T_{\mu\nu}(\phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi.$$

Let  $f$  be a function of  $r^*$  and consider a vector field of the form

$$V = f(r^*) \frac{\partial}{\partial r^*}, \quad (2)$$

for an arbitrary function  $f$ .

Let the function  $\beta$  be defined by

$$\beta = \frac{1-\mu}{r} - \frac{x}{\alpha^2 + x^2},$$

where

$$x = r^* - \alpha - \alpha^{\frac{1}{2}}$$

for a (sufficiently large) constant  $\alpha$  to be determined later. Define the currents

$$\begin{aligned} J_\mu^{V,0}(\phi) &= T_{\mu\nu}(\phi) V^\nu, \\ J_\mu^{V,1}(\phi) &= T_{\mu\nu}(\phi) V^\nu + \frac{1}{4} \left( f' + 2 \frac{1-\mu}{r} f \right) \partial_\mu(\phi)^2 - \frac{1}{4} \partial_\mu \left( f' + 2 \frac{1-\mu}{r} f \right) \phi^2, \\ J_\mu^{V,2}(\phi) &= T_{\mu\nu}(\phi) V^\nu + \frac{1}{4} \left( f' + 2 \frac{1-\mu}{r} f \right) \partial_\mu(\phi)^2 - \frac{1}{4} \partial_\mu \left( f' + 2 \frac{1-\mu}{r} f \right) \phi^2 \\ &\quad - \frac{1}{2} \frac{f'}{f(1-\mu)} \beta V_\mu \phi^2, \end{aligned}$$

and the divergences

$$K^{V,i} = \nabla^\mu J_\mu^{V,i}.$$

We compute

$$\begin{aligned} K^{V,0}(\phi) &= \frac{f'(\partial_{r^*}\phi)^2}{1-\mu} + |\nabla\phi|^2 \left( \frac{\mu'}{2(1-\mu)} + \frac{1-\mu}{r} \right) f \\ &\quad - \frac{1}{4} \left( 2f' + 4 \frac{1-\mu}{r} f \right) \phi^\alpha \phi_\alpha, \end{aligned} \quad (3)$$

$$\begin{aligned} K^{V,1}(\phi) &= \frac{f'}{1-\mu} (\partial_{r^*}\phi)^2 + |\nabla\phi|^2 \left( \frac{\mu'}{2(1-\mu)} + \frac{1-\mu}{r} \right) f \\ &\quad - \frac{1}{4} \left( \square \left( f' + 2 \frac{1-\mu}{r} f \right) \right) \phi^2 \\ &= \frac{f'}{1-\mu} (\partial_{r^*}\phi)^2 + |\nabla\phi|^2 \left( \frac{\mu'}{2(1-\mu)} + \frac{1-\mu}{r} \right) f \\ &\quad - \frac{1}{4} \left( \frac{1}{1-\mu} f''' + \frac{4}{r} f'' - \frac{4\mu'}{r(1-\mu)} f' + \frac{2}{(1-\mu)r} \left( \frac{\mu'(1-\mu)}{r} - \mu'' \right) f \right) \phi^2, \end{aligned}$$

$$\begin{aligned}
K^{V,2}(\phi) &= \frac{f'}{(1-\mu)} (\partial_{r^*} \phi + \beta \phi)^2 + \frac{r-3M}{r^2} f |\nabla \phi|^2 \\
&\quad - \frac{1}{4} \frac{1}{1-\mu} \left( f''' + \frac{4f''x}{\alpha^2 + x^2} + \frac{4\alpha^2 f'}{(\alpha^2 + x^2)^2} \right) \phi^2 - \frac{\mu f}{2r^3} (4\mu - 3) \phi^2.
\end{aligned}$$

## 2 Definition of the current $J_\mu$

Let  $\Omega_i$ ,  $i = 1, \dots, 3$  denote a basis of angular momentum operators.

Define

$$f^a = -\frac{C_*}{\alpha^2 r^2}$$

for a constant  $C_*$ , dependent on  $\alpha$ , both of which to be determined in what follows,

$$f^b = \frac{1}{\alpha} \left( \tan^{-1} \frac{x}{\alpha} - \tan^{-1}(-1 - \alpha^{-\frac{1}{2}}) \right), \quad (f^b)' = \frac{1}{\alpha^2 + x^2}$$

$$X^a = f^a \partial_{r^*}$$

$$X^b = f^b \partial_{r^*}$$

and let

$$\begin{aligned}
J &= J^{X^a,0}(\phi) + \sum_{i=1}^3 J^{X^b,1}(\Omega_i \phi), \\
K &= \nabla^\mu J_\mu.
\end{aligned}$$

Note that  $J$  and  $K$  both depend on the 2-jet of  $\phi$ .

## 3 Nonnegativity

**Proposition 3.1.** *For the  $K$  defined above with  $\alpha$ ,  $C_*$  suitably chosen,*

$$\int_{\mathbb{S}^2} K r^2 dA_{\mathbb{S}^2} \geq 0.$$

*Proof.* Note that  $(f^a)' \geq 0$ ,  $(f^b)' \geq 0$  and  $2(f^a)' + 4\frac{1-\mu}{r}f^a = 0$ . We then conclude that

$$\begin{aligned}
K &\geq \sum_{i=1}^3 \left( f^b \frac{r-3M}{r^2} |\nabla \Omega_i \phi|^2 + \left( f^b \frac{\mu(3-4\mu)}{2r^3} + F \right) (\Omega_i \phi)^2 \right) - 2C_* \frac{r-3M}{\alpha^2 r^4} |\nabla \phi|^2 \\
&\quad (4)
\end{aligned}$$

where

$$F := -\frac{1}{4} \frac{1}{1-\mu} \left( (f^b)''' + \frac{4(f^b)''x}{\alpha^2 + x^2} + \frac{4\alpha^2 (f^b)'}{(\alpha^2 + x^2)^2} \right) = \frac{1}{2(1-\mu)} \frac{x^2 - \alpha^2}{(x^2 + \alpha^2)^3}.$$

Note that  $\sum_{i=1}^3 (\Omega_i \phi)^2 = r^2 |\nabla \phi|^2$ . Note also the Poincaré inequality for  $\Omega_i \phi$

$$2 \int_{\mathbb{S}^2} (\Omega_i \phi)^2 r^2 dA_{\mathbb{S}^2} \leq r^2 \int_{\mathbb{S}^2} |\nabla \Omega_i \phi|^2 r^2 dA_{\mathbb{S}^2}. \quad (5)$$

Thus, to prove the proposition, in view of (4), it suffices to show

$$2f^b \frac{r-3M}{r^4} + \left( f^b \frac{\mu(3-4\mu)}{2r^3} + F \right) - 2C_* \frac{r-3M}{\alpha^2 r^6} \geq 0. \quad (6)$$

or alternatively,

$$2f^b(r-3M)r^2 + H(r) + Fr^6 \geq 0 \quad (7)$$

where

$$H(r) := f^b \frac{\mu(3-4\mu)}{2} r^3 - 2C_* \frac{r-3M}{\alpha^2} = r^4 \left( f^b \frac{\mu(3-4\mu)}{2r^3} r^2 - 2C_* \frac{r-3M}{\alpha^2 r^4} \right).$$

Note that the first term on the right hand side of (7) is manifestly nonnegative.

We first establish that the function  $H$  satisfies  $H(r) \geq 0$ . Observe that  $H > 0$  in the region  $r \leq 8M/3$  and that  $H(3M) = 0$ . With the constant  $C_*$  chosen such that  $\frac{dH}{dr}|_{r=3M} = 0$ , in order to show that  $H \geq 0$  in the range  $8M/3 \leq r < \infty$ , it suffices to show that there exists an  $R$  such that  $H \geq 0$  for  $r \geq R$ , and  $\frac{d^2 H}{dr^2} \geq 0$  in  $8M/3 \leq r \leq R$ .

We compute

$$\frac{dH}{dr} = M(f^b)' \frac{3r^2 - 8Mr}{1 - \mu} + 2Mf^b(3r - 4M) - 2C_* \alpha^{-2}.$$

Setting  $\frac{dH}{dr}|_{r=3M} = 0$  implies

$$C_* = \alpha^2 M(f^b)' \frac{3r^2 - 8Mr}{2(1 - \mu)}|_{r=3M} = \frac{9\alpha^2 M^3}{2((\alpha + \alpha^{\frac{1}{2}})^2 + \alpha^2)}.$$

Given  $\alpha$ , let this then be our choice of  $C_*$ . It is clear that as  $\alpha \rightarrow \infty$ ,

$$C_* \rightarrow \frac{9M^3}{4}.$$

Note that from the definition of  $f^b$  we have that there exist constants  $c, R$ , independent of  $\alpha$ , such that for all values  $r \geq R$

$$f^b \geq \frac{c}{\alpha} \min \left\{ \frac{r}{\alpha}, 1 \right\}. \quad (8)$$

As a consequence, taking into account the value of the constant  $C_*$ , we see that for  $r \geq R$ ,

$$H(r) \geq \frac{cM}{\alpha} (3r^2 - 8Mr) \min \left\{ \frac{r}{\alpha}, 1 \right\} - 2C_* \frac{r-3M}{\alpha^2} > 0,$$

To show  $H \geq 0$ , it remains to check that  $\frac{d^2 H}{dr^2} \geq 0$  for all  $r \in [\frac{8M}{3}, R]$ .

We compute

$$\frac{d^2 H}{dr^2} = M(f^b)'' \frac{3r^2 - 8Mr}{(1 - \mu)^2} + 4M(f^b)' \frac{3r - 4M}{1 - \mu} - 2M^2(f^b)' \frac{3r^2 - 8Mr}{r^2(1 - \mu)^2} + 6Mf^b.$$

First, we easily see that for  $r \in [\frac{8M}{3}, R]$ ,

$$M(f^b)'' \frac{3r^2 - 8Mr}{(1 - \mu)^2} \leq C\alpha^{-3},$$

for a  $C$  independent of  $\alpha$ . As a consequence, this term will be dominated (for sufficiently large  $\alpha$  by the other terms in the expression for  $\frac{d^2 H}{dr^2}$ , which are of the order of  $\alpha^{-2}$ .

We combine the terms containing  $(f^b)'$ , taking into account that for  $r \in [\frac{8M}{3}, R]$  we have  $(1 - \mu) \geq \frac{1}{4}$ , to obtain

$$\begin{aligned} 4M(f^b)' \frac{3r - 4M}{1 - \mu} - 2M^2(f^b)' \frac{3r - 8M}{r(1 - \mu)^2} &= \frac{2M(f^b)'}{r(1 - \mu)} \left( 6r^2 - 4Mr - M \frac{3r - 8M}{1 - \mu} \right) \\ &\geq \frac{2M(f^b)'}{r(1 - \mu)} (6r^2 - 16Mr + 32M^2) \geq c\alpha^{-2}. \end{aligned}$$

Moreover, in the region  $r \in [3M, R]$  the last term  $6Mf^b$  is non-negative, which immediately implies desired conclusion that  $\frac{d^2 H}{dr^2} > 0$ . On the other hand, for  $r \in [\frac{8M}{3}, 3M]$  we have

$$f^b = \int_r^{3M} \frac{(f^b)'}{1 - \mu} dr \geq \frac{(f^b)'|_{r=3M}}{1 - \mu} (r - 3M).$$

From the expression for  $(f^b)' = (\alpha^2 + x^2)^{-1}$  with  $x = r^* - \alpha - \alpha^{\frac{1}{2}}$  we easily see that for all  $r \in [\frac{8M}{3}, 3M]$

$$(f^b)'|_{r=3M} = (f^b)'(r) + O(\alpha^{-3}).$$

Therefore,

$$\frac{d^2 H}{dr^2} \geq \frac{2M(f^b)'}{r(1 - \mu)} (9r^2 - 25Mr + 32M^2) + O(\alpha^{-3}) \geq c\alpha^{-2}$$

for  $r \in [\frac{8M}{3}, 3M]$ .

To prove (7), and thus Proposition 3.1 it now suffices to establish the inequality

$$2f^b \frac{r - 3M}{r^4} + F \geq 0.$$

We note that the function  $F$  is non-negative outside the region  $-\alpha < x < \alpha$ . Thus, in view of the nonnegativity of the first term above, it follows that (7) holds for  $x \notin [-\alpha, \alpha]$ .

For  $x \in [-\alpha, \alpha]$  on the other hand, we have that  $r^* \in [\alpha^{\frac{1}{2}}, 2\alpha + \alpha^{\frac{1}{2}}]$ , which implies that, for  $\alpha$  sufficiently large,

$$r \geq c\alpha^{\frac{1}{2}}, \quad r = \left(1 + O(\alpha^{-\frac{1}{2}})\right)(x + \alpha + \alpha^{\frac{1}{2}}), \quad \mu = O(\alpha^{-\frac{1}{2}}).$$

We may approximate functions  $F$  and  $2f^b \frac{r-3M}{r^4}$  by the expressions

$$F \sim \frac{1}{2} \frac{x^2 - \alpha^2}{(x^2 + \alpha^2)^3}, \quad 2f^b \frac{r-3M}{r^4} \sim \frac{2(x + \alpha)}{(x^2 + \alpha^2)(x + \alpha + \alpha^{\frac{1}{2}})^3}.$$

It suffices then to establish the bound

$$\frac{(\alpha - x)(x + \alpha + \alpha^{\frac{1}{2}})^3}{4(x^2 + \alpha^2)^2} < \frac{9}{10}, \quad \forall x \in [-\alpha, \alpha].$$

For  $-\alpha \leq x \leq 0$  we have

$$(\alpha - x) \leq 2\alpha, \quad (x + \alpha + \alpha^{\frac{1}{2}})^3 < \frac{3}{2}\alpha^3, \quad (x^2 + \alpha^2) \geq \alpha^2,$$

where the middle inequality follows if the constant  $\alpha$  is chosen to be sufficiently large. Therefore,

$$\frac{(\alpha - x)(x + \alpha + \alpha^{\frac{1}{2}})^3}{4(x^2 + \alpha^2)^2} < \frac{3}{4}.$$

On the other hand, for  $0 \leq x \leq \alpha$  we have

$$(x + \alpha + \alpha^{\frac{1}{2}})^3 < 2^{\frac{3}{2}} \frac{8}{7} (x^2 + \alpha^2)^{\frac{3}{2}}, \quad \alpha - x \leq \alpha.$$

Thus,

$$\frac{(\alpha - x)(x + \alpha + \alpha^{\frac{1}{2}})^3}{4(x^2 + \alpha^2)^2} < \frac{2}{7} 2^{\frac{3}{2}} < \frac{9}{10}.$$

□

## 4 Quantities controlled

**Proposition 4.1.** *There exists a constant  $C$  depending only on  $M$  such that*

$$C \int_{\mathbb{S}^2} K r^2 dA_{\mathbb{S}^2} \geq \int_{\mathbb{S}^2} \left( \frac{1}{r^3} (\partial_{r^*} \phi)^2 + \frac{(r-3M)^2}{r} |\nabla^2 \phi|^2 + \frac{r^3}{(1-\mu)(|r^*|+1)^4} |\nabla \phi|^2 \right) r^2 dA_{\mathbb{S}^2}.$$

*Proof.* Revisit the proof of Proposition 3.1 and recall the nonnegative quantities that were dropped. □

Now define  $X^{\text{aux}} = r^{-3} \partial_{r^*}$  and define  $J_\mu^{\text{aux}} = J_\mu^{X^{\text{aux}}, 0}$ ,  $K^{\text{aux}} = \nabla^\mu J_\mu^{\text{aux}}$ . We easily see (cf. Section 7.4 of [4])

**Proposition 4.2.** *There exists a constant  $C$  depending only on  $M$  such that*

$$\begin{aligned} & C \int_{\mathbb{S}^2} (K^{\text{aux}} + K) r^2 dA_{\mathbb{S}^2} \\ & \geq \int_{\mathbb{S}^2} \left( \frac{1}{r^3} (\partial_{r^*} \phi)^2 + \frac{(r-3M)^2}{r} |\nabla^2 \phi|^2 + \frac{r^3}{(1-\mu)(|r^*|+1)^4} |\nabla \phi|^2 + \frac{1}{r^4} (\partial_t \phi)^2 \right) r^2 dA_{\mathbb{S}^2}. \end{aligned}$$

## 5 Boundary terms

To turn Propositions 4.1, 4.2 into estimates with the help of the divergence theorem, we need to understand the boundary terms arising from the integration of the currents  $K$ ,  $K^{\text{aux}}$ .

**Proposition 5.1.** *Let  $\mathcal{S}$  be an achronal subset of  $\text{clos}(\mathcal{D})$  such that  $\text{clos}(\mathcal{S}) \cap \mathcal{H}^- = \emptyset$ . There exist constants  $\epsilon, C > 0$ , depending only on  $\mathcal{S}$  such that*

$$\left| \int_{\mathcal{S}} J_{\mu}^{\text{aux}} n^{\mu} \right| + \left| \int_{\mathcal{S}} J_{\mu} n^{\mu} \right| \leq \int_{\mathcal{S}} \left( C J_{\mu}^T(\phi) + \epsilon J_{\mu}^Y(\phi) + C \sum_i J_{\mu}^T(\Omega_i \phi) \right) n^{\mu}$$

where  $J_{\mu}^T = T^{\nu} T_{\mu\nu}$ ,  $J_{\mu}^Y = Y^{\nu} T_{\mu\nu}$  where  $T$  denotes the Killing field  $\frac{\partial}{\partial t}$  and  $Y$  denotes the local observer vector field of [3]. In the case where  $\mathcal{S}$  is spacelike,  $n^{\mu}$  denotes the future pointing unit normal and the measure is the volume form. In the case where  $\mathcal{S}$  is null,  $n^{\mu}$  and the measure are defined appropriately.

If  $\phi_t$  denotes the one-parameter group of transformations generated by  $T$ , then one can take  $C(\mathcal{S}, \epsilon) = C(\phi_t(\mathcal{S}), \epsilon)$ , and  $\epsilon(\mathcal{S}, C) = \epsilon(\phi_t(\mathcal{S}), C)$ . If  $\mathcal{S}$  is a constant  $t$ -surface or if  $\text{clos}(\mathcal{S}) \cap (\mathcal{H}^+ \cup \mathcal{H}^-) = \emptyset$ , then the  $J^Y$  term can be omitted from the right hand side, i.e. one can take  $\epsilon = 0$ . If  $\mathcal{S}$  is a constant  $u$ -surface such that  $r^* \leq r_0^*$ ,  $t \geq t_0$  in  $\mathcal{S}$ , then as  $r_0^* \rightarrow -\infty$  with  $t_0$  fixed one can take  $C$  uniformly bounded and  $\epsilon \rightarrow 0$ .

## 6 The estimates

Consider a “trapezoidal” region  $\mathcal{R}$  defined by the inequalities  $t_1 \leq t \leq t_2$ ,  $r_1^* - (t_2 - t) \leq r^* \leq r_2^* + (t_2 - t)$ . Let  $\mathcal{S}_1$  denote the timelike past boundary  $\{t_1\} \times [r_1^* - (t_2 - t_1), r_2^* + (t_2 - t_1)]$ , and let  $\mathcal{F}_1$  denote the constant- $t - r^*$  boundary. Propositions 5.1, 4.1, 4.2, the divergence theorem and the fact that  $K^T = 0$  give

**Theorem 6.1.** *There exists a  $C$  depending only on  $M$  and an  $\epsilon$  depending only on  $r_1^*$  with  $\epsilon \rightarrow 0$  as  $r_1^* \rightarrow -\infty$  such that*

$$\begin{aligned} \int_{\mathcal{R}} \left( \frac{1}{r^3} (\partial_{r^*} \phi)^2 + \frac{(r - 3M)^2}{r} |\nabla^2 \phi|^2 + \frac{r^3}{(1 - \mu)(|r^*| + 1)^4} |\nabla \phi|^2 + \frac{1}{r^4} (\partial_t \phi)^2 \right) \\ \leq C \int_{\mathcal{S}_1} \left( J_{\mu}^T(\phi) + \sum_i J_{\mu}^T(\Omega_i \phi) \right) n^{\mu} + \epsilon \int_{\mathcal{F}_1} J_{\mu}^Y(\phi) n^{\mu}. \end{aligned}$$

Let  $\mathcal{D}$  denote the domain of outer communications. Note that taking  $r_1^* \rightarrow -\infty$ ,  $t_2 \rightarrow \infty$ , an immediate corollary of the above is

**Corollary 6.1.** *There exists a constant  $C$  depending only on  $M$  such that*

$$\begin{aligned} \int_{\mathcal{D}} \left( \frac{1}{r^3} (\partial_{r^*} \phi)^2 + \frac{(r - 3M)^2}{r} |\nabla^2 \phi|^2 + \frac{r^3}{(1 - \mu)(|r^*| + 1)^4} |\nabla \phi|^2 + \frac{1}{r^4} (\partial_t \phi)^2 \right) \\ \leq C \int_{t=0} \left( J_{\mu}^T(\phi) + \sum_i J_{\mu}^T(\Omega_i \phi) \right) n^{\mu}. \end{aligned}$$



On the other hand, using the  $Y$  estimate of [3] one obtains

**Theorem 6.2.** *There exists a constant  $C$  depending only on  $M$  such that*

$$\begin{aligned} \int_{\mathcal{R}} \left( \frac{1}{r^3} (\partial_{r^*} \phi)^2 + \frac{(r-3M)^2}{r} |\nabla^2 \phi|^2 + \frac{r^3}{(1-\mu)(|r^*|+1)^4} |\nabla \phi|^2 + \frac{1}{r^4} (\partial_t \phi)^2 \right) \\ \leq C \int_{\mathcal{S}_1} \left( J_\mu^{T+Y}(\phi) + \sum_i J_\mu^T(\Omega_i \phi) \right) n^\mu. \end{aligned}$$

The above Theorem can be used in conjunction with the  $Y$  estimates and the Morawetz vector field  $u^2 \partial_u + v^2 \partial_v$  to obtain the results of [3]. Note that, in contrast to the scheme of [3], the vector field  $u^2 \partial_u + v^2 \partial_v$  is not necessary to control the boundary terms arising from  $J^\mu$ .

## 7 Comments

As noted in the introduction, by a simple perturbation argument, our results also apply to remove reference to spherical harmonics from the arguments of [4] in the case of small  $M\sqrt{\Lambda}$ . This relies on the fact that error terms in the immediate vicinity of the horizons can be absorbed via the use of the  $Y$  and  $\bar{Y}$  estimates.

It is also interesting to note that the currents based on the vector fields  $f_\ell \partial_{r^*}$  of [3] were also essential even to obtain just the uniform boundedness statement  $|\phi| \leq C$ , where  $C$  is a constant depending only on a norm of initial data. Recall that this statement was originally proven by Kay and Wald [5],<sup>2</sup> with methods that relied heavily on the staticity of exterior Schwarzschild and a certain discrete isometry of the maximally extended solution. In view of the results of the present paper, uniform boundedness can now be shown without the Kay-Wald trick, without the vector field  $u^2 \partial_u + v^2 \partial_v$  and without recourse to spherical harmonics.

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## References

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<sup>2</sup>In spite of the classical paper [5], many later authors continue to use the term “linear stability” or “decay” to refer to results which are completely compatible with the statement  $\sup_{\mathcal{D}} |\phi| = \infty$  where  $\mathcal{D}$  denotes the domain of outer communications. It is hard even to imagine what kind of stability such results would indicate.

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